Lecture 22

Thursday, 23 April 2026   9:49 AM

Problem: n agents N, m goods G, v_{ij} is value
allocation $A = (A_1, \dots, A_n)$ to maximize $\left(\prod_{i=1}^{n} v_i(A_i)\right)^{1/n}$
This is the Nash welfare

Easy to show problem is NP-hard, also APX-hard w/ an apx. factor ~ 1.061

Will give an $e^{1/e}+\varepsilon \sim 1.445$ approx algo, using LP-rounding.
Based on paper by Feng, Li, 2025
Result shown earlier by Barman, Krishnamurthy, Vaish 2018
(In fact showed a stronger result)

Defn.: EF1 alloc.
An alloc $A = (A_1, \dots, A_n)$ is envy free upto one good (EF1) if, whenever $v_i(A_j) > v_i(A_i)$,
$\exists g \in A_j : v_i(A_j \setminus g) \leq v_i(A_i)$

Thm: For additive valuations, a max NSW alloc is EF1
(w/o proof)
(Caragiannis et al. 2019)

Thm: For agents w/ identical, additive valuations, any EF1 allocation is an $e^{1/e}$ approx to max NSW

(will use this later).

Natural LP: $\max \prod v_i(S) \quad \dots$
Take log: $\max \frac{1}{n}\sum_i \ln\left(\sum_j v_{ij} x_{ij}\right) \quad \dots$

Write down configuration LP:

CONF-LP

$$\max \frac{1}{n}\sum_{S \subseteq G} y_{i,S} \ln v_i(S)$$
$$\text{s.t.} \quad \sum_{S \subseteq G} y_{i,S} \leq 1 \quad \forall i$$
$$\sum_{S: j \in S}\sum_i y_{i,S} \leq 1 \quad \forall g$$
$$y_{i,S} \geq 0$$

Lemma: Given $\varepsilon > 0$, can in time poly$(m, n, \frac{1}{\varepsilon})$ obtain a soln to LP-CONF that (i) has value $\geq OPT - \ln(1+\varepsilon)$ and (ii) is supported on poly$(m, n, \frac{1}{\varepsilon})$ variables
(later)

Assume we have such a soln., & further that
$\forall j,\ \sum_{i,\, S: j \in S} y_{i,S} = 1$, & $\forall i,\ \sum_S y_{i,S} = 1$.

Define $x_{ij} = \sum_{S: j \in S} y_{i,S}$. $\quad p_i = \lceil \sum_j x_{ij} \rceil$.

Draw the b.p. graph & fractional matching x' as earlier:

Goods: o o o ⋮ o   Agents: o o ⋮ o } p_1 o ⋮ o } p_2 ⋮   o ⋮ o } p_n

For edges, & fractional matching x':

Fix agent i. Assume $v_{i1} \geq v_{i2} \geq \dots \geq v_{im}$
Let j_1 be (smallest index) s.t. $\sum_{j \leq j_1} x_{ij} \geq 1$, j_2 be s.t. $\sum_{j \leq j_2} x_{ij} \geq 2$, ... j_{p_i} be largest index s.t. $x_{ij} > 0$.
Let i_1 be first copy of agent i, ..., i_{p_i} be last copy.
For i_1: for $j < j_1$ s.t. $x_{ij} > 0$: $x'(i_1, j) = x_{ij}$
for $j = j_1$: $x'(i_1, j_1) = 1 - \sum_{j < j_1} x_{ij}$

For i_2: for $j = j_1$, $x'(i_2, j_1) = \sum_{j \leq j_1} x_{ij} - 1$
for $j_1 < j < j_2$, $x'(i_2, j) = x_{ij}$
for $j = j_2$, $x'(i_2, j_2) = 2 - \sum_{j < j_2} x_{ij}$

and so on for $i_3, \dots, i_{p_i}$.

EXAMPLE: 2 agents, 4 goods a, b, c, d

valuations:

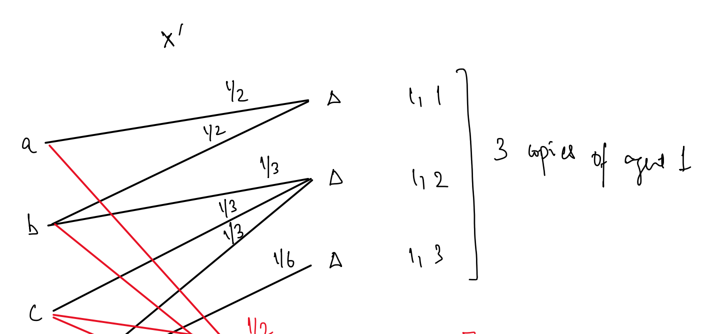
	a	b	c	d
agent 1	4	3	2	1
2	1	2	3	4

Suppose the LP returns: $y_{1,\{ab\}} = 1/2$, $y_{1,\{bcd\}} = 1/3$, $y_{1,\{d\}} = 1/6$
& $y_{2,\{ab\}} = 1/6$, $y_{2,\{acd\}} = 1/3$, $y_{2,\{c\}} = 1/3$, $y_{2,\{d\}} = 1/6$

Then x:

	a	b	c	d	
agent 1	1/2	5/6	1/3	1/2	$p_1 = \lceil 2\frac{1}{6} \rceil = 3$
2	1/2	1/6	2/3	1/2	$p_2 = \lceil 1\frac{5}{6} \rceil = 2$

Graph & x':
(note that agents have opp. orders on the goods,
1: a > b > c > d
2: d > b > c > a)



	a	b	c	d
1,1	1/2	1/2		
1,2		1/3	1/3	1/3
1,3				1/6
2,1			1/2	1/2
2,2	1/2	1/6	1/6	

x' can be obtained as the foll. convex combination of integral matchings:

$\frac{1}{6} \times$ [a 1,1; b 1,2; c 2,1; d 1,3] $+ \frac{1}{6} \times$ [a 1,1; b 1,2; c 2,2; d 2,1] $+ \frac{1}{6} \times$ [a 1,1; b 2,2; c 1,2; d 2,1]
$+ \frac{1}{6} \times$ [a 2,2; b 1,1; c 1,2; d 2,1] $+ \frac{1}{3} \times$ [a 2,2; b 1,1; c 2,1; d 1,2]

Now x' satisfies the following:
(1) $\forall i, j \quad \sum_{s \leq p_i} x'(i_s, j) = x_{ij}$
hence $\forall j,\ \sum_i \sum_{s \leq p_i} x'(i_s, j) = \sum_i x_{ij} = 1$

(2) $\forall i,\ s < p_i,\ \sum_j x'(i_s, j) = 1$
$\sum_j x'(i_{p_i}, j) \leq 1$

(3) $\forall i,\ s < s' \leq p_i$ suppose $x'(i_s, j) > 0$ & $x'(i_{s'}, j') > 0$
then $v_{ij} \geq v_{ij'}$.

Lemma: Given x', can in poly time find matchings $M^1, \dots, M^k$ & convex coeffs $\lambda_1 \dots \lambda_k$ s.t. $k \leq (m \times \sum_i p_i) + 1$, and for any edge (i_s, j), $\sum_{r=1}^{k} \lambda_r M^r_{i_s, j} = x'_{i_s, j}$
(prove yourself)

Claim: Consider matchings $M, M' \in \{M^1, \dots, M^k\}$ & let A, A' be the corresponding alloc. Then if $v_i(M) > v_i(M')$ then for some $j \in A$, $v_i(M \setminus j) \leq v_i(M')$

Proof: Note that M & M' both saturate $p_i - 1$ copies of agent i.
Thus $|M|, |M'| \in \{p_i, p_i - 1\}$. Assume $|M| = p_i$, $|M'| = p_i - 1$.
Let $j_1 \dots j_{p_i}$ be goods in decreasing order of value assigned to $i_1 \dots i_{p_i}$ in M, & $j_1' \dots j'_{p_i - 1}$ be goods in decreasing order of value assigned to $i_1 \dots i_{p_i - 1}$ in M'. Then $v_i(j_1') \geq v_i(j_2), \dots v_i(j'_{p_i - 1}) \geq v_i(j_{ip})$
Hence $v_i(M') \geq v_i(M \setminus j_1)$ ∎

Algo: form the b.p graph, obtain x', $M^1, \dots, M^k$ as given by the lemma.
Find the matching $M \in \{M^1, \dots, M^k\}$ that maximizes NSW, return it.

For analysis will show that for a matching M drawn randomly from $M^1 \dots M^k$ w.p. $\lambda_1 \dots \lambda_k$ satisfies

$$\mathbb{E}\left[\sum_{i \in N} \ln v_i(M)\right] \geq \sum_{i, S \subseteq G} y_{i,S} \ln(v_i(S)) - 1/e$$

Thus clearly this holds for the best choice of M.

Lemma: For every agent i, for a randomly chosen M, let M_i be the set of items assigned to i. Then
$$\mathbb{E}[\ln v_i(M_i)] \geq \sum_{S \subseteq G} y_{i,S} \ln v_i(S) - 1/e$$

Proof: Choose Δ large enough so that:
(i) $\forall r = 1 \dots k$, $\Delta \lambda_r \in \mathbb{Z}_+$,
(ii) $\forall S$, $\Delta y_{i,S} \in \mathbb{Z}_+$

Create a new instance, w/ Δ copies of agent i, & $\Delta x'_{ij}$ copies of each item j.

Consider the following two allocs.
(1) Alloc A:
For $r = 1 \dots k$, $\Delta\lambda_r$ agents get the bundle of goods M_i^r.
check that this is an allocation: total agents allocated $= \sum_{r=1}^{k} \Delta\lambda_r = \Delta$
for each good j, # of agents allocated to $= \sum_{r=1}^{k} \Delta\lambda_r \cdot \mathbb{1}_{j \text{ alloc to } i \text{ in } M^r} = \Delta \sum_{r=1}^{k} \lambda_r M^r_{ij}$
$= \Delta x'_{ij}$

Claim: Alloc A is EF1
(proof done earlier)
Hence, $\left(\prod_{r=1}^{k} v_i(M^r)^{\Delta\lambda_r}\right)^{1/\Delta} \geq \max NW \cdot e^{-1/e}$

(2) Alloc B:
Assign $\Delta y_{i,S}$ agents the bundle S.
Can check this is valid alloc.
Then $\prod_{r=1}^{k} v_i(M^r)^{\lambda_r} \geq e^{-1/e}\left(\prod_S v_i(S)^{\Delta y_{i,S}}\right)^{1/\Delta}$
$= e^{-1/e}\left(\prod_S v_i(S)^{y_{i,S}}\right)$

Taking logs, $\sum_{r=1}^{k} \lambda_r \ln(v_i(M^r)) \geq \sum_S y_{i,S} \ln(v_i(S)) - \frac{1}{e}$
or $\mathbb{E}[\ln v_i(M)] \geq \sum_S y_{i,S} \ln v_i(S) - \frac{1}{e}$ ∎

Then $\mathbb{E}\left[\frac{1}{n}\sum_i \ln v_i(M)\right] \geq \frac{1}{n}\sum_{i,S} y_{i,S} \ln v_i(S) - 1/e$

or $\mathbb{E}\left[\prod_i v_i(M)^{1/n}\right] \geq e^{-1/e} \exp\left(\frac{1}{n}\sum_{i,S} y_{i,S} \ln v_i(S)\right)$
$\geq e^{-1/e} \cdot \frac{OPT}{1+\varepsilon}$